Partial Fractions

① $\int \frac{1}{x^2-1}\,dx$

$$\frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$$

$1 = A(x-1) + B(x+1)$

$1 = Ax - A + Bx + B$

$1 = (A+B)x + (B-A)$

$B-A=1 \qquad A+B=0$

$B=1+A \qquad A+1+A=0$

$B=1-\frac{1}{2} \qquad 2A=-1$

$B=\frac{1}{2} \qquad A=-\frac{1}{2}$

$\int \frac{-\frac{1}{2}}{x+1}\,dx + \int \frac{\frac{1}{2}}{x-1}\,dx$

$-\frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C$

$\frac{1}{2}\left[\ln|x-1| - \ln|x+1|\right] + C$

$$\boxed{\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C}$$

③ $\int \frac{3}{x^2+x-2}\,dx$

$$\frac{3}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

$3 = A(x-1) + B(x+2)$

$3 = Ax - A + Bx + 2B$

$3 = (A+B)x + (-A+2B)$

$-A+2B=3 \qquad A+B=0$

$-(-B)+2B=3 \qquad A=-B$

$3B=3 \qquad A=-1$

$B=1$

$\int \frac{-1}{x+2}\,dx + \int \frac{1}{x-1}\,dx$

$-\ln|x+2| + \ln|x-1| + C$

$$\boxed{\ln\left|\frac{x-1}{x+2}\right| + C}$$

⑤ $\int \frac{5-x}{2x^2+x-1}$ $\frac{5-x}{(2x-1)(x+1)} = \frac{A}{2x-1} + \frac{B}{x+1}$

$5-x = A(x+1) + B(2x-1)$

$5-x = Ax + A + 2Bx - B$

$x(A+2B) + (A-B)$

$A+2B=-1$	$A-B=5$
$B+5+2B=-1$	$A=B+5$
$3B=-6$	$A=-2+5$
$B=-2$	$A=3$

$\int \frac{3}{2x-1} dx + \frac{-2}{x+1}$

$3\int \frac{1}{2x-1} dx$

$u=2x-1$
$du=2dx$
$\frac{1}{2}du = dx$

$\boxed{\frac{3}{2} \ln|2x-1| - 2\ln|x+1| + C}$

⑦ $\int \frac{x^2+12x+12}{x^3-4x} dx$ $\frac{x^2+12x+12}{x(x+2)(x-2)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2}$
  $\underbrace{}_{x(x^2-4)}$

$x^2+12x+12 = A(x+2)(x-2) + Bx(x-2) + Cx(x+2)$

$Ax^2 - 4A + Bx^2 - 2Bx + Cx^2 + 2Cx$

$x^2+12x+12 = (A+B+C)x^2 + (2C-2B)x - 4A$

$A+B+C=1$ $2C-2B=12$ $-4A=12$

$-3+B+C=1$ $\boxed{A=-3}$

$B+C=4$

$C=4-B$ $-2B+2C=12$ $A+B+C=1$

$-2B+2(4-B)=12$ $-3-1+C=1$

$-2B+8-2B=12$ $\boxed{C=5}$

$-4B=4$

$\boxed{B=-1}$

$\int \frac{-3}{x} dx + \int \frac{-1}{x+2} dx + \int \frac{5}{x-2} dx$

$\boxed{-3\ln|x| - \ln|x+2| + 5\ln|x-2| + C}$

⑨ $\int \frac{2x^3 - 4x^2 - 15x + 5}{x^2 - 2x - 8} dx$

$$x^2 - 2x - 8 \overline{) \begin{array}{c} 2x + 0 + \frac{x+5}{x^2-2x-8} \\ 2x^3 - 4x^2 - 15x + 5 \end{array}}$$

$\underline{\pm (-2x^3 \pm 4x^2 \mp 16x)}$

$x + 5$

$\int 2x \, dx + \int \frac{x+5}{(x-4)(x+2)} dx$

$\frac{x+5}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2}$

$x + 5 = A(x+2) + B(x-4)$

$x + 5 = Ax + 2A + Bx - 4B$

$(A+B)x + (2A - 4B)$

$A + B = 1$	$2A - 4B = 5$
$A = 1 - B$	$2(1-B) - 4B = 5$
$A = 1 - -\frac{1}{2}$	$2 - 2B - 4B = 5$
$A = \frac{3}{2}$	$-6B = 3$
	$B = -\frac{1}{2}$

$\int 2x \, dx + \int \frac{3/2}{x-4} dx + \int \frac{-1/2}{x+2} dx$

$\boxed{x^2 + \frac{3}{2} \ln |x-4| - \frac{1}{2} \ln |x+2| + C}$